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VIBRATIONS OF DOUBLY-ROTATED-CUT QUARTZ PLATES WITH MONOCLINIC SYMMETRY†

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Abstract—For certain doubly rotated cuts of quartz, the elastic stiffness constants have the same symmetry and absolute values as those for certain rotated-*Y*-cuts; but 4 of the 13 constants have signs reversed. Mathematical solutions of corresponding problems for the two types of cut have the same form but the numerical results may be the same or different according as the constants with changed signs enter the solution as even or odd powers or products. Examples of both are exhibited.

1. EQUATIONS FOR DOUBLY-ROTATED-CUT QUARTZ PLATES

Alpha-quartz has an axis of three-fold symmetry, say X_3, and three axes of two-fold symmetry one of which is designated as X_1 in a right-handed, rectangular coordinate system X_i, $i = 1, 2, 3$. A doubly rotated set of axes is obtained by rotating the X_i system a positive angle θ about X_1 and a positive angle ϕ about X_3 to a new orientation x_i. The direction cosines l_{ij}, of the x_i axes with respect to the X_i axes are

$$\begin{array}{llll} & X_1 & X_2 & X_3 \\ x_1 & l_{11} = \cos\phi & l_{12} = \sin\phi & l_{13} = 0 \\ x_2 & l_{21} = -\sin\phi\cos\theta & l_{22} = \cos\phi\cos\theta & l_{23} = \sin\theta \\ x_3 & l_{31} = \sin\phi\sin\theta & l_{32} = -\cos\phi\sin\theta & l_{33} = \cos\theta. \end{array} \quad (1)$$

Rotated-*Y*-cut and doubly-rotated-cut plates are cut with faces perpendicular to x_2, as shown in Fig. 1.

The elastic stiffness constants c_{rstu}, $r, s, t, u = 1, 2, 3$ (or, in the reduced indicial notation: c_{pq}, $p, q = 1 \ldots 6$), referred to the rotated axes x_i, are expressed in terms of the constants c^0_{ijkl}, referred to the X_i, by

$$c_{rstu} = c^0_{ijkl} l_{ri} l_{sj} l_{tk} l_{ul} \quad (2)$$

summed over $i, j, k, l = 1, 2, 3$.

Of the 21 possible constants $c^0_{pq} (= c^0_{qp})$, referred to the X_i coordinates, only six are independent inasmuch as, for α-quartz[1],

$$c^0_{22} = c^0_{11},\ c^0_{55} = c^0_{44},\ c^0_{23} = c^0_{13},\ c^0_{14} = c^0_{56} = -c^0_{24},\ c^0_{66} = \tfrac{1}{2}(c^0_{11} - c^0_{12}),$$
$$c^0_{15} = c^0_{25} = c^0_{35} = c^0_{45} = c^0_{16} = c^0_{26} = c^0_{36} = c^0_{46} = c^0_{34} = 0. \quad (3)$$

The values of the remaining c^0_{pq}, as given by Bechmann[2], are

$$\begin{array}{ll} c^0_{11} = 86.74 & c^0_{12} = 6.98 \\ c^0_{33} = 107.2 & c^0_{13} = 11.91 \\ c^0_{44} = 57.94 & c^0_{14} = -17.91 \end{array} \quad (4)$$

in units of 10^{10} dyn/cm^2 or 10^9 N/m^2.

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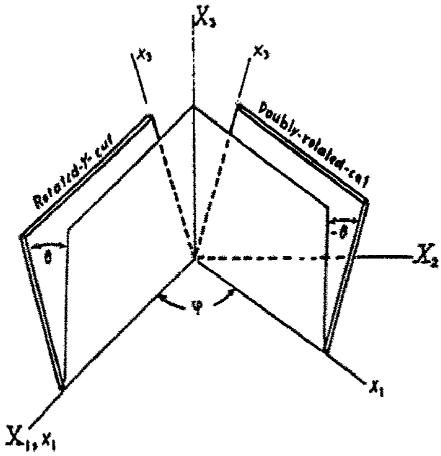

Fig. 1. Rotated-Y-cut and doubly-rotated-cut quartz plates. X_1 and X_3 are digonal and trigonal axes of symmetry, respectively.

From (2) and (3) we have

$$\begin{aligned} c_{rstu} = {} & c^0_{11}\,[l_{r1}l_{s1}l_{t1}l_{u1} + l_{r2}l_{s2}l_{t2}l_{u2} + \tfrac{1}{2}(l_{r1}l_{s2} + l_{r2}l_{s1})(l_{t1}l_{u2} + l_{t2}l_{u1})] \\ & + c^0_{33}l_{r3}l_{s3}l_{t3}l_{u3} \\ & + c^0_{44}[(l_{r2}l_{s3} + l_{r3}l_{s2})(l_{t2}l_{u3} + l_{t3}l_{u2}) + (l_{r3}l_{s1} + l_{r1}l_{s3})(l_{t3}l_{u1} + l_{t1}l_{u3})] \\ & + c^0_{12}\,[l_{r1}l_{s1}l_{t2}l_{u2} + l_{r2}l_{s2}l_{t1}l_{u1} - \tfrac{1}{2}(l_{r1}l_{s2} + l_{r2}l_{s1})(l_{t1}l_{u2} + l_{t2}l_{u1})] \\ & + c^0_{13}[l_{r3}l_{s3}(l_{t1}l_{u1} + l_{t2}l_{u2}) + l_{t3}l_{u3}(l_{r1}l_{s1} + l_{r2}l_{s2})] \\ & + c^0_{14}[(l_{r2}l_{s3} + l_{r3}l_{2})(l_{t1}l_{u1} - l_{t2}l_{u2}) + (l_{t2}l_{u3} + l_{t3}l_{u2})(l_{r1}l_{s1} - l_{r2}l_{s2}) \\ & + (l_{r1}l_{s2} + l_{r2}l_{s1})(l_{t3}l_{u1} + l_{t1}l_{u3}) + (l_{t1}l_{u2} + l_{t2}l_{u1})(l_{r3}l_{s1} + l_{r1}l_{s3})]. \end{aligned} \tag{5}$$

Finally, upon substituting (1) in (5), we find

$$c_{11} = c^0_{11}$$

$$c_{12} = c^0_{12}\cos^2\theta + c^0_{13}\sin^2\theta + c^0_{14}\sin 2\theta\cos 3\phi$$

$$c_{13} = c^0_{12}\sin^2\theta + c^0_{13}\cos^2\theta - c^0_{14}\sin 2\theta\cos 3\phi$$

$$c_{14} = (c^0_{13} - c^0_{12})\sin\theta\cos\theta + c^0_{14}\cos 2\theta\cos 3\phi$$

$$c_{15} = c^0_{14}\cos\theta\sin 3\phi$$

$$c_{16} = c^0_{14}\sin\theta\sin 3\phi$$

$$c_{22} = c^0_{11}\cos^4\theta + c^0_{33}\sin^4\theta + (c^0_{44} + \tfrac{1}{2}c^0_{13})\sin^2 2\theta + 4\,c^0_{14}\sin\theta\cos^3\theta\cos 3\phi$$

$$c_{23} = \tfrac{1}{4}(c^0_{11} + c^0_{33} - 2\,c^0_{13} - 4\,c^0_{44})\sin^2 2\theta + c^0_{13} + \tfrac{1}{2}c^0_{14}\sin 4\theta\cos 3\phi$$

$$\begin{aligned} c_{24} = {} & -\,c^0_{11}\sin\theta\cos^3\theta + c^0_{33}\sin^3\theta\cos\theta + \tfrac{1}{4}(c^0_{44} + \tfrac{1}{2}c^0_{13})\sin 4\theta \\ & -\,c^0_{14}\cos\theta\cos 3\theta\cos 3\phi \end{aligned}$$

$$c_{25} = c^0_{14}\,(3\sin^2\theta - 1)\cos\theta\sin 3\phi$$

$$c_{26} = 3\,c^0_{14}\sin\theta\cos^2\theta\sin 3\phi$$

$$c_{33} = c^0_{11}\sin^4\theta + c^0_{33}\cos^4\theta + (c^0_{44} + \tfrac{1}{2}c^0_{13})\sin^2 2\theta + 4\,c^0_{14}\sin^3\theta\cos\theta\cos 3\phi$$

$$\begin{aligned} c_{34} = {} & -\,c^0_{11}\sin^3\theta\cos\theta + c^0_{33}\sin\theta\cos^3\theta - \tfrac{1}{4}(c^0_{44} + \tfrac{1}{2}c^0_{13})\sin 4\theta \\ & -\,c^0_{14}\sin\theta\sin 3\theta\cos 3\phi \end{aligned}$$

$$
\begin{aligned}
c_{35} &= -3\, c_{14}^0 \sin^2\theta \cos\theta \sin 3\phi \\
c_{36} &= c_{14}^0 (3\cos^2\theta - 1)\sin\theta \sin 3\phi \\
c_{44} &= \tfrac{1}{4}(c_{11}^0 + c_{33}^0 - 2\, c_{13}^0 - 4\, c_{44}^0)\sin^2 2\theta + c_{44}^0 + \tfrac{1}{2}c_{14}^0 \sin 4\theta \cos 3\phi \\
c_{45} &= c_{14}^0 (3\cos^2\theta - 1)\sin\theta \sin 3\phi \\
c_{46} &= c_{14}^0 (3\sin^2\theta - 1)\cos\theta \sin 3\phi \\
c_{55} &= \tfrac{1}{2}(c_{11}^0 - c_{12}^0)\sin^2\theta + c_{44}^0 \cos^2\theta - c_{14}^0 \sin 2\theta \cos 3\phi \\
c_{56} &= -\tfrac{1}{4}(c_{11}^0 - c_{12}^0 - 2\, c_{44}^0)\sin 2\theta + c_{14}^0 \cos 2\theta \cos 3\phi \\
c_{66} &= \tfrac{1}{2}(c_{11}^0 - c_{12}^0)\cos^2\theta + c_{44}^0 \sin^2\theta + c_{14}^0 \sin 2\theta \cos 3\phi
\end{aligned}
\tag{6}
$$

These are the c_{pq} which appear in the stress–strain relations referred to the x_i:

$$T_{ij} = c_{ijkl}S_{kl} \quad \text{or} \quad T_p = c_{pq}S_q \tag{7}$$

in which the strains, S_{ij} or S_p, in terms of displacements, u_i, are

$$
\begin{aligned}
S_{11} &= S_1 = u_{1,1} \qquad & 2\,S_{23} &= S_4 = u_{3,2} + u_{2,3} \\
S_{22} &= S_2 = u_{2,2} \qquad & 2\,S_{31} &= S_5 = u_{1,3} + u_{3,1} \\
S_{33} &= S_3 = u_{3,3} \qquad & 2\,S_{12} &= S_6 = u_{2,1} + u_{1,2}.
\end{aligned}
\tag{8}
$$

Upon substituting (8) in (7) and the result in the stress-equations of motion:

$$T_{ij,i} = \rho \ddot{u}_j, \tag{9}$$

we find the displacement-equations of motion:

$$D_{ij}u_j = \rho \ddot{u}_j \tag{10}$$

or

$$
\begin{aligned}
D_{11}u_1 + D_{12}u_2 + D_{13}u_3 &= \rho \ddot{u}_1, \\
D_{21}u_1 + D_{22}u_2 + D_{23}u_3 &= \rho \ddot{u}_2, \\
D_{31}u_1 + D_{32}u_2 + D_{33}u_3 &= \rho \ddot{u}_3,
\end{aligned}
\tag{11}
$$

in which the $D_{ij}(= D_{ji})$ are the differential operators

$$
\begin{aligned}
D_{11} &= c_{11}\partial_1^2 + c_{66}\partial_2^2 + c_{55}\partial_3^2 + 2\, c_{56}\partial_2\partial_3 + 2\, c_{15}\partial_3\partial_1 + 2\, c_{16}\partial_1\partial_2, \\
D_{22} &= c_{22}\partial_2^2 + c_{44}\partial_3^2 + c_{66}\partial_1 + 2\, c_{64}\partial_3\partial_1 + 2\, c_{26}\partial_1\partial_2 + 2\, c_{24}\partial_2\partial_3, \\
D_{33} &= c_{33}\partial_3^2 + c_{55}\partial_1^2 + c_{44}\partial_2^2 + 2\, c_{45}\partial_1\partial_2 + 2\, c_{34}\partial_2\partial_3 + 2\, c_{35}\partial_3\partial_1, \\
D_{23} &= c_{56}\partial_1^2 + c_{24}\partial_2^2 + c_{34}\partial_3^2 + (c_{23} + c_{44})\partial_2\partial_3 + (c_{36} + c_{45})\partial_3\partial_1 + (c_{25} + c_{64})\partial_1\partial_2, \\
D_{31} &= c_{64}\partial_2^2 + c_{35}\partial_3^2 + c_{15}\partial_1^2 + (c_{31} + c_{55})\partial_3\partial_1 + (c_{14} + c_{56})\partial_1\partial_2 + (c_{36} + c_{45})\partial_2\partial_3, \\
D_{12} &= c_{45}\partial_3^2 + c_{16}\partial_1^2 + c_{26}\partial_2^2 + (c_{12} + c_{66})\partial_1\partial_2 + (c_{25} + c_{64})\partial_2\partial_3 + (c_{14} + c_{56})\partial_3\partial_1,
\end{aligned}
\tag{12}
$$

where $\partial_i\partial_j = \partial^2/\partial x_i \partial x_j$, $\partial_i^2 = \partial^2/\partial x_i^2$.

For traction-free planes parallel to the coordinate planes, it is required that on x_1 = constant:

$$
\begin{aligned}
T_{11} &= T_1 = c_{11}S_1 + c_{12}S_2 + c_{13}S_3 + c_{14}S_4 + c_{15}S_5 + c_{16}S_6 = 0,\\
T_{12} &= T_6 = c_{61}S_1 + c_{62}S_2 + c_{63}S_3 + c_{64}S_4 + c_{65}S_5 + c_{66}S_6 = 0,\\
T_{13} &= T_5 = c_{51}S_1 + c_{52}S_2 + c_{53}S_3 + c_{54}S_4 + c_{55}S_5 + c_{56}S_6 = 0;
\end{aligned}
\tag{13}
$$

on x_2 = constant:

$$
\begin{aligned}
T_{21} &= T_6 = c_{61}S_1 + c_{62}S_2 + c_{63}S_3 + c_{64}S_4 + c_{65}S_5 + c_{66}S_6 = 0,\\
T_{22} &= T_2 = c_{21}S_1 + c_{22}S_2 + c_{23}S_3 + c_{24}S_4 + c_{25}S_5 + c_{26}S_6 = 0,\\
T_{23} &= T_4 = c_{41}S_1 + c_{42}S_2 + c_{43}S_3 + c_{44}S_4 + c_{45}S_5 + c_{46}S_6 = 0;
\end{aligned}
\tag{14}
$$

on x_3 = constant:

$$
\begin{aligned}
T_{31} &= T_5 = c_{51}S_1 + c_{52}S_2 + c_{53}S_3 + c_{54}S_5 + c_{55}S_5 + c_{56}S_6 = 0,\\
T_{32} &= T_4 = c_{41}S_1 + c_{42}S_2 + c_{43}S_3 + c_{44}S_4 + c_{45}S_5 + c_{46}S_6 = 0,\\
T_{33} &= T_3 = c_{31}S_1 + c_{32}S_2 + c_{33}S_3 + c_{34}S_4 + c_{35}S_5 + c_{36}S_6 = 0.
\end{aligned}
\tag{15}
$$

2. ROTATED-Y-CUTS VS 60° DOUBLY-ROTATED-CUTS

Case A. If $\phi = 0$, $\theta \neq 0$ (the rotated-Y-cuts) then $\sin 3\phi = 0$, $\cos 3\phi = 1$ and, from (6),

$$
c_{15} = c_{16} = c_{25} = c_{26} = c_{35} = c_{36} = c_{45} = c_{46} = 0. \tag{16}
$$

The remaining 13 constants are those for monoclinic symmetry with x_1 the digonal axis.

Case B. If $\phi = 60°$, $\theta \neq 0$ (doubly-rotated cuts), then $\sin 3\phi = 0$, $\cos 3\phi = -1$ and (16) again holds so that the symmetry is the same as for rotated-Y-cuts. Even if θ is the same in A and B, all the surviving constants in B (except for c_{11} which remains fixed) are different from the corresponding ones in A as the last term in each c_{pq} has its sign reversed. However, if θ in B is the negative of θ in A, nine of the constants are the same for the two cuts and the remaining four have the same absolute values in A and B but are of opposite sign.

To summarize the properties of the two sets of constants $c_{pq}(\phi, \theta)$:

$$
\begin{aligned}
c_{pq}(0, \theta) &= c_{pq}(60°, -\theta) \text{ for } pq = 11, 12, 13, 22, 23, 33, 44, 55, 66,\\
c_{pq}(0, \theta) &= -c_{pq}(60°, -\theta) \text{ for } pq = 14, 24, 34, 56.
\end{aligned}
\tag{17}
$$

The displacement equations of motion reduce to

$$
\begin{aligned}
&(c_{11}\partial_1^2 + c_{66}\partial_2^2 + c_{55}\partial_3^2 + 2\, c_{56}\partial_2\partial_3)u_1 + [(c_{12} + c_{66})\partial_1\partial_2 + (c_{14} + c_{56})\partial_3\partial_1]u_2\\
&\qquad + [(c_{13} + c_{55})\partial_3\partial_1 + (c_{14} + c_{56})\partial_1\partial_2]u_3 = \rho\ddot{u}_1,\\
&[(c_{12} + c_{66})\partial_1\partial_2 + (c_{14} + c_{56})\partial_3\partial_1]u_1 + (c_{22}\partial_2^2 + c_{44}\partial_3^2 + c_{66}\partial_1^2 + 2\, c_{24}\partial_2\partial_3)u_2\\
&\qquad + [c_{56}\partial_1^2 + c_{24}\partial_2^2 + c_{34}\partial_3^2 + (c_{23} + c_{44})\partial_2\partial_3]u_3 = \rho\ddot{u}_2,\\
&[(c_{12} + c_{66})\partial_1\partial_2 + (c_{14} + c_{56})\partial_3\partial_1]u_1 + [c_{56}\partial_1^2 + c_{24}\partial_2^2 + c_{34}\partial_3^2 + (c_{23} + c_{44})\partial_2\partial_3]u_2\\
&\qquad + (c_{33}\partial_3^2 + c_{55}\partial_1^2 + c_{44}\partial_2^2 + 2\, c_{34}\partial_2\partial_3)u_3 = \rho\ddot{u}_3,
\end{aligned}
\tag{18}
$$

and the conditions (13), (14), (15) for traction-free boundaries reduce to:

on x_1 = constant:

$$\begin{aligned} T_{11} &= T_1 = c_{11}u_{1,1} + c_{12}u_{2,2} + c_{13}u_{3,3} + c_{14}(u_{3,2} + u_{2,3}) = 0, \\ T_{12} &= T_6 = c_{56}(u_{1,3} + u_{3,1}) + c_{66}(u_{2,1} + u_{1,2}) = 0, \\ T_{13} &= T_5 = c_{55}(u_{1,3} + u_{3,1}) + c_{56}(u_{2,1} + u_{1,2}) = 0; \end{aligned} \tag{19}$$

on x_2 = constant:

$$\begin{aligned} T_{21} &= T_6 = c_{56}(u_{1,3} + u_{3,1}) + c_{66}(u_{2,1} + u_{1,2}) = 0, \\ T_{22} &= T_2 = c_{12}u_{1,1} + c_{22}u_{2,2} + c_{23}u_{3,3} + c_{24}(u_{3,2} + u_{2,3}) = 0, \\ T_{23} &= T_4 = c_{14}u_{1,1} + c_{24}u_{2,2} + c_{34}u_{3,3} + c_{44}(u_{3,2} + u_{2,3}) = 0; \end{aligned} \tag{20}$$

on x_3 = constant:

$$\begin{aligned} T_{31} &= T_5 = c_{55}(u_{1,3} + u_{3,1}) + c_{56}(u_{2,1} + u_{1,2}) = 0, \\ T_{32} &= T_4 = c_{14}u_{1,1} + c_{24}u_{2,2} + c_{34}u_{3,3} + c_{44}(u_{3,2} + u_{2,3}) = 0, \\ T_{33} &= T_3 = c_{13}u_{1,1} + c_{23}u_{2,2} + c_{33}u_{3,3} + c_{34}(u_{3,2} + u_{2,3}) = 0. \end{aligned} \tag{21}$$

Any solution of the equations of motion (and boundary conditions, if any) referred to rotated axes with $\phi = 0$, $\theta = \theta'$ is the same solution, at least in form, referred to axes with $\phi = 60°$, $\theta = -\theta'$. Whether or not the solutions are the same numerically depends on the occurrence of c_{14}, c_{24}, c_{34}, c_{56} as even or odd powers or products in the resulting formulas.

In the following sections, we review solutions obtained previously for $c_{pq}(0, \theta)$ and determine if the transition to $c_{pq}(60°, -\theta)$ changes the numerical results.

3. PLANE WAVES IN A PLATE

In a plate with faces at $x_2 = \pm b$, we consider waves propagating in the direction of the two-fold axis of symmetry x_1:

$$\begin{aligned} u_1 &= A_1 \sin(\xi_2 x_2 + \xi_3 x_3) \sin(\xi_1 x_1 - \omega t), \\ u_2 &= -A_2 \cos(\xi_2 x_2 + \xi_3 x_3) \cos(\xi_1 x_1 - \omega t), \\ u_3 &= -A_3 \cos(\xi_2 x_2 + \xi_3 x_3) \cos(\xi_1 x_1 - \omega t). \end{aligned} \tag{22}$$

Upon substituting (22) in (19) and setting the determinant of the coefficients of the A_j equal to zero, we find the equation

$$|\lambda_{ij} - \delta_{ij}V^2| = 0, \qquad \lambda_{ij} = \lambda_{ji}, \tag{23}$$

in which δ_{ij} is the Kronecker delta,

$$\begin{aligned} \lambda_{11} &= \bar{c}_{11} + \beta^2 + \bar{c}_{55}\Gamma^2 + 2\,\bar{c}_{56}\beta\Gamma, & \lambda_{23} &= \bar{c}_{56} + \bar{c}_{24}\beta^2 + \bar{c}_{34}\Gamma^2 + (\bar{c}_{23} + \bar{c}_{44})\beta\Gamma, \\ \lambda_{22} &= 1 + \bar{c}_{22}\beta^2 + \bar{c}_{44}\Gamma^2 + 2\,\bar{c}_{24}\beta\Gamma, & \lambda_{31} &= (\bar{c}_{14} + \bar{c}_{56})\beta + (\bar{c}_{13} + \bar{c}_{55})\Gamma, \\ \lambda_{33} &= \bar{c}_{55} + \bar{c}_{44}\beta^2 + \bar{c}_{33}\Gamma^2 + 2\,\bar{c}_{34}\beta\Gamma, & \lambda_{12} &= (1 + \bar{c}_{12})\beta + (\bar{c}_{14} + \bar{c}_{56})\Gamma. \end{aligned} \tag{24}$$

$$\bar{c}_{pq} = c_{pq}/c_{66} \qquad \beta = \xi_2/\xi_1, \qquad \Gamma = \xi_3/\xi_1, \qquad V^2 = \Omega^2/\bar{\xi}_1^2 = \rho\omega^2/c_{66}\xi_1^2. \tag{25}$$

In (25), β and Γ are the ratios of the wave length along x_1 to the wave lengths along x_2 and x_3, respectively; V is the ratio of the velocity to the velocity $v = (c_{66}/\rho)^{1/2}$; $\bar{\xi}_1(= 2\xi_1 b/\pi)$ is the ratio of the thickness, $2b$, of the plate to the half-wave-length along x_1; and Ω is the ratio of the circular frequency ω to the frequency $\pi v/2b$.

For given β and Γ, (23) is a bicubic in the velocity ratio V:

$$V^6 + BV^4 + CV^2 + D = 0, \tag{26}$$

in which

$$\begin{aligned} B &= -(\lambda_{11} + \lambda_{22} + \lambda_{33}), \\ C &= \lambda_{22}\lambda_{33} + \lambda_{33}\lambda_{11} + \lambda_{11}\lambda_{22} - \lambda_{23}^2 - \lambda_{31}^2 - \lambda_{12}^2, \\ D &= \lambda_{11}\lambda_{23}^2 + \lambda_{22}\lambda_{31}^2 + \lambda_{33}\lambda_{12}^2 - \lambda_{11}\lambda_{22}\lambda_{33} - 2\,\lambda_{23}\lambda_{31}\lambda_{12}. \end{aligned} \tag{27}$$

The coefficients of the bicubic are different for $c_{14}, c_{24}, c_{34}, c_{56}$ positive and negative. Hence, for given β and Γ, the roots of (26) yield different sets of velocity ratios V_1, V_2, V_3 for the rotated-Y-cut with $c_{pq}(0, \theta)$ and the doubly-rotated-cut with $c_{pq}(60°, -\theta)$. An example is illustrated in Fig. 2 in which either β is the abscissa and $\Gamma = 10$ or vice versa. In either case, the lowest velocity ratios V_3 exhibit little difference for the two cuts—and this is the branch which would contribute predominantly to the fundamental thickness-shear mode of the plate. However, the differences for the upper velocities are larger, at least for large β and Γ—as much as 13% for $\beta = \Gamma = 10$. These differences survive any boundary conditions that may be applied.

4. EKSTEIN'S SOLUTION

It will be observed, in Fig. 2, that the velocity ratios are the same for Case A and Case B if Γ (or β) is zero. This is the situation for modes with straight crests along x_3 (or x_2). In the case $\Gamma = 0$, λ_{23} and λ_{31} change sign, in the passage from Case A to Case B, but they enter the coefficients of the bicubic (26) only as their product and as squares—resulting in no change in roots. To examine whether this persists after the introduction of free faces of the plate, we consider Ekstein's solution[3] for modes with straight crests along x_3 in a plate with free faces on $x_2 = \pm b$.

With $\Gamma = 0$, and fixed ξ_1 and V, (23) yields three roots β_n^2, $n = 1, 2, 3$. Thus, for steady state vibrations, (22) may be written as

$$\begin{aligned} u_1 &= \sum_{n=1}^{3} A_{1n} \sin \xi_1\beta_n x_2 \sin \xi_1 x_1 \, e^{i\omega t}, \\ u_2 &= -\sum_{n=1}^{3} A_{2n} \cos \xi_1\beta_n x_2 \cos \xi_1 x_1 \, e^{i\omega t}, \\ u_3 &= -\sum_{n=1}^{3} A_{3n} \cos \xi_1\beta_n x_2 \cos \xi_1 x_1 \, e^{i\omega t}. \end{aligned} \tag{28}$$

Then the boundary conditions (20):

$$T_{2j} = 0, \quad j = 1, 2, 3, \quad \text{on} \quad x_2 = \pm b, \tag{29}$$

result in Ekstein's frequency equation; which may be written in the form[4]

$$|\,\mu_{in}\,| = 0, \qquad i, n = 1, 2, 3, \tag{30}$$

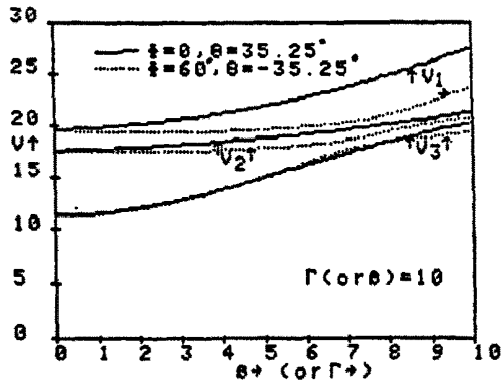

Fig. 2. Comparison of wave velocities for $c_{pq}(0, \theta)$ and $c_{pq}(60°, -\theta)$ as functions of the ratios, β and Γ, of wave lengths in the x_2 and x_3 directions to the wave length in the direction x_1 of the wave normal.

where

$$\mu_{1n} = (\beta_n L_{1n} + L_{2n} + \bar{c}_{56}L_{3n}) \cot \beta_n \xi_1 b,$$
$$\mu_{2n} = \bar{c}_{12}L_{1n} + \beta_n(\bar{c}_{22}L_{2n} + \bar{c}_{24}L_{3n}), \tag{31}$$
$$\mu_{3n} = \bar{c}_{14}L_{1n} + \beta_n(\bar{c}_{24}L_{2n} + \bar{c}_{44}L_{3n}),$$

$$L_{in} = \operatorname{cof}(\lambda_{ni} - \delta_{ni}V_n^2)/\operatorname{cof}(\lambda_{nn} - V_n^2). \tag{32}$$

In the passage from $c_{pq}(0, \theta)$ to $c_{pq}(60°, -\theta)$, the L_{in} and, hence, the μ_{in} (which depend on c_{14}, c_{24}, c_{56}) change sign for subscripts 13, 31, 23, 32 while the remaining terms in (31) and (32) do not change. But those μ_{in} which do change appear only as product pairs in (30) and, hence, the roots of (30) do not change. These roots are usually depicted graphically as a many branched dispersion relation between Ω (as ordinate) and $\bar{\xi}_1$ (as abscissa):

$$\Omega = \Omega(\bar{\xi}_1) \tag{33}$$

as illustrated in [4]. Alternatively, the abscissa could be $1/\bar{\xi}_1$:

$$\hat{\Omega} = \hat{\Omega}(1/\bar{\xi}_1). \tag{34}$$

Suppose the plate has additional bounding planes $x_1 = \pm a$ at which the conditions are uniformly point-mixed, e.g. vanishing u_2, T_{11}, T_{13} corresponding to "simply supported" in the elementary theory of flexural vibrations of plates. For real roots of (30), these conditions are satisfied by $\xi_1 = m\pi/2a$, where m is an even integer; so that, for real roots, the dispersion relation converts to

$$\hat{\Omega} = \hat{\Omega}(a/mb). \tag{35}$$

Elimination of m from the abscissa requires only that each branch of the dispersion relation (35) be replaced by a sequence of branches obtained by multiplication of its abscissa by a sequence of integers. In this way, the branches of the dispersion relation for the infinite plate are converted to the branches of the frequency spectrum, Ω vs a/b, of the "simply supported" plate. As the process does not involve c_{14}, c_{24} and c_{56} anew, the frequency spectrum is not altered by a change of $c_{pq}(0, \theta)$ to $c_{pq}(60°, -\theta)$.

There is no closed solution of the three-dimensional equations for the case of *free* boundaries at $x_1 = \pm a$ and the situation there is not obvious inasmuch as c_{14} and c_{56}

enter into the traction-free conditions

$$T_{11} = T_{12} = T_{13} = 0 \quad \text{on} \quad x_1 = \pm a \tag{36}$$

as may be seen in (19).

5. EFFECT OF FREE EDGES

As a substitute for the unavailable extension of Ekstein's solution of the three-dimensional equations to accommodate a pair of parallel, free edges, there exists a solution of two-dimensional approximate equations[5]. For the case of straight crested flexural waves travelling in the direction of x_1 in a plate with free faces at $x_2 = \pm b$, the three dimensional displacements are approximated by

$$u_1 = x_2\psi_1(x_1)e^{i\omega t}, \qquad u_2 = U_2(x_1)e^{i\omega t}, \qquad u_3 = U_3(x_1)e^{i\omega t} \tag{37}$$

and the differential equations governing them are

$$\begin{aligned} \kappa\, c_{56}U_{3,11} + \kappa^2\, c_{66}(U_{2,11} + \psi_{1,1}) &= -\rho\omega^2 U_2, \\ c_{55}U_{3,11} + \kappa\, c_{56}(U_{2,11} + \psi_{1,1}) &= -\rho\omega^2 U_3, \\ \gamma_{11}\psi_{1,11} - 3b^{-2}[\kappa\, c_{56}U_{3,1} + \kappa^2\, c_{66}(U_{2,1} + \psi_1)] &= -\rho\omega^2\psi_1, \end{aligned} \tag{38}$$

where

$$\kappa^2 = \pi^2/12, \qquad \gamma_{11} = c_{11} - c_{12}^2/c_{22} - (c_{14} - c_{12}c_{24})^2/(c_{44} - c_{24}^2/c_{22}). \tag{39}$$

There is no change of sign of γ_{11} with change of sign of c_{14} and c_{24}; so only c_{56}, in (38), changes sign with the passage from $c_{pq}(0, \theta)$ to $c_{pq}(60°, -\theta)$.

The displacements are taken as

$$U_2 = A_2 b \sin \xi x_1, \qquad U_3 = A_3 b \sin \xi x_1, \qquad \psi_1 = A_4 \cos \xi x_1. \tag{40}$$

Then, from (38),

$$\begin{aligned} (\hat{\xi}^2 - 3\Omega^2)A_2 + \hat{c}_{56}\hat{\xi}^2 A_3 + \hat{\xi}A_4 &= 0, \\ \hat{c}_{56}\hat{\xi}^2 A_2 + (\hat{c}_{55}\hat{\xi}^2 - 3\Omega^2)A_3 + \hat{c}_{56}\hat{\xi}A_4 &= 0, \\ \hat{\xi}A_2 + \hat{c}_{56}\hat{\xi}A_3 + (\hat{\gamma}_{11}\hat{\xi}^2 + 1 - \Omega^2)A_4 &= 0, \end{aligned} \tag{41}$$

where

$$\hat{\xi} = \xi b, \qquad \hat{c}_{55} = c_{55}/\kappa^2 c_{66}, \qquad \hat{c}_{56} = c_{56}/\kappa\, c_{66}, \qquad \hat{\gamma}_{11} = \gamma_{11}/3\kappa^2 c_{66}. \tag{42}$$

The determinant of the coefficients of the A_i in (41), set equal to zero, is the equation

$$\begin{aligned} &\hat{\gamma}_{11}(\hat{c}_{55} - \hat{c}_{56}^2)\hat{\xi}^6 - \Omega^2[3\hat{\gamma}_{11}(1 + \hat{c}_{55}) + \hat{c}_{55} - \hat{c}_{56}^2]\hat{\xi}^4 \\ &\quad + 3\Omega^2[\Omega^2 - \hat{c}_{55}(1 - \Omega^2) + 3\hat{\gamma}_{11}\Omega^2 + \hat{c}_{56}^2]\hat{\xi}^2 + 9\Omega^4(1 - \Omega^2) = 0, \end{aligned} \tag{43}$$

which, for a fixed frequency ratio Ω, is a bicubic in $\hat{\xi}^2$ whose roots are independent of change of sign of c_{14}, c_{24}, c_{56}. Thus, as in the three-dimensional case, the dispersion relation does not change with passage from $c_{pq}(0, \theta)$ to $c_{pq}(60°, -\theta)$.

For each Ω, (43) has three roots $\hat{\xi}_n^2$, $n = 1, 2, 3$, and (41) has three sets of amplitude ratios $A_2:A_3:A_4$. Let $\bar{A}_n$, $n = 1, 2, 3$, be the value of A_4 for the nth root $\hat{\xi}_n^2$; and let

$$\alpha_{2n} = \frac{A_2}{A_4} = \hat{\xi}_n(\hat{c}_{56}^2\hat{\xi}_n^2 + 3\,\Omega^2 - \hat{c}_{55}\hat{\xi}_n^2)/\Delta_n,$$
$$\alpha_{3n} = \frac{A_3}{A_4} = 3\,\hat{c}_{56}\hat{\xi}_n\,\Omega^2/\Delta_n, \tag{44}$$
$$\Delta_n = (\hat{\xi}_n^2 - 3\,\Omega^2)(\hat{c}_{55}\hat{\xi}_n^2 - 3\,\Omega^2) - \hat{c}_{56}^2\hat{\xi}_n^4$$

for each root $\hat{\xi}_n^2$. Then (40) may be written as

$$U_2 = b\sum_{n=1}^{3}\bar{A}_n\alpha_{2n}\sin\xi_n x_1,$$
$$U_3 = b\sum_{n=1}^{3}\bar{A}_n\alpha_{3n}\sin\xi_n x_1, \tag{45}$$
$$\psi_1 = \sum_{n=1}^{3}\bar{A}_n\cos\xi_n x_1.$$

The conditions for free edges at $x_1 = \pm a$ are: the horizontal and vertical shears, N_5 and Q_1, and the bending moment, M_1, vanish. Thus, on $x_1 = \pm a$,

$$N_5 = 2b[c_{55}U_{3,1} + \kappa\, c_{56}(U_{2,1} + \psi_1)] = 0,$$
$$Q_1 = 2b\kappa[c_{56}U_{3,1} + \kappa\, c_{66}(U_{2,1} + \psi_1)] = 0, \tag{46}$$
$$M_1 = (2b^3/3)\gamma_{11}\psi_{1,1} = 0.$$

Upon substituting (45) into (46), we obtain

$$\sum_{n=1}^{3}\bar{A}_n\bar{\alpha}_{1n}\cos\xi_n a = 0,$$
$$\sum_{n=1}^{3}\bar{A}_n\bar{\alpha}_{2n}\cos\xi_n a = 0, \tag{47}$$
$$\sum_{n=1}^{3}\bar{A}_n\xi_n\sin\xi_n a = 0,$$

where

$$\bar{\alpha}_{1n} = c_{55}\alpha_{3n}\hat{\xi}_n + \kappa\, c_{56}(\alpha_{2n}\hat{\xi}_n + 1),$$
$$\bar{\alpha}_{2n} = c_{56}\alpha_{3n}\hat{\xi}_n + \kappa\, c_{66}(\alpha_{2n}\hat{\xi}_n + 1). \tag{48}$$

The frequency equation is obtained by setting the determinant of the coefficients of the $\bar{A}_n$ in (47) equal to zero:

$$\tilde{A}_1\tan\xi_1 a + \tilde{A}_2\tan\xi_2 a + \tilde{A}_3\tan\xi_3 a = 0 \tag{49}$$

where

$$\tilde{A}_1 = \hat{\xi}_1(\bar{\alpha}_{12}\bar{\alpha}_{23} - \bar{\alpha}_{22}\bar{\alpha}_{13}),$$
$$\tilde{A}_2 = \hat{\xi}_2(\bar{\alpha}_{13}\bar{\alpha}_{21} - \bar{\alpha}_{23}\bar{\alpha}_{11}), \tag{50}$$
$$\tilde{A}_3 = \hat{\xi}_3(\bar{\alpha}_{11}\bar{\alpha}_{22} - \bar{\alpha}_{21}\bar{\alpha}_{12}).$$

Upon substituting (44) in (48) and the result in (50), the frequency equation (49) becomes

$$\hat{\xi}_1\Delta_1(\hat{\xi}_2^2 - \hat{\xi}_3^2)\tan\xi_1 a + \hat{\xi}_2\Delta_2(\hat{\xi}_3^2 - \hat{\xi}_1^2)\tan\xi_2 a + \hat{\xi}_3\Delta_3(\hat{\xi}_1^2 - \hat{\xi}_2^2)\tan\xi_3 a = 0: \quad (51)$$

an equation which does not change when $c_{pq}(0, \theta)$ is replaced by $c_{pq}(60°, -\theta)$.

6. VIBRATIONS OF A STRIP

An exact solution of the three-dimensional equations exists for coupled thickness-twist and face-shear modes of vibration in a rotated-Y-cut strip with a parallelogrammic cross-section and all four faces free of traction[6]. The displacements are $u_2 = u_3 = 0$ and, omitting a factor $e^{i\omega t}$,

$$\begin{aligned} u_1 = {} & A\sin\xi_2 x_2 \cos\xi_3(\bar{c}_{56}x_2 - x_3) + B\sin\xi_2 x_2 \sin\xi_3(\bar{c}_{56}x_2 - x_3) \\ & + C\cos\xi_2 x_2\cos\xi_3(\bar{c}_{56}x_2 - x_3) + D\cos\xi_2 x_2 \sin\xi_3(\bar{c}_{56}x_2 - x_3), \end{aligned} \quad (52)$$

where $\bar{c}_{56} = c_{56}/c_{66}$, as before in (25).

The equations of motion (18) are satisfied if

$$\rho\omega^2 = c_{66}\xi_2^2 + \gamma_{55}\xi_3^2, \; \gamma_{55} = c_{55} - c_{56}^2/c_{66}; \quad (53)$$

and the faces at $x_2 = \pm b$ satisfy the traction-free conditions (20) if

$$2\xi_2 b = m\pi \quad (54)$$

where m is an odd integer for solutions A and B and an even integer for solutions C and D.

A pair of planes parallel to the x_1-axis, making dihedral angles α with the $x_1 - x_2$ plane and distant $2c\cos\alpha$ part, as illustrated in Fig. 3, are free of traction if

$$\alpha = \arctan \bar{c}_{56} \quad (55)$$

and

$$2\xi_3 c = n\pi \quad (56)$$

where n is an even integer for solutions A and C and an odd integer for solutions B and D.

The frequencies are

$$\omega = \frac{m\pi}{2b}\left(\frac{c_{66}}{\rho}\right)^{1/2}\left(1 + \frac{n^2\gamma_{55}b^2}{m^2 c_{66}c^2}\right)^{1/2}. \quad (57)$$

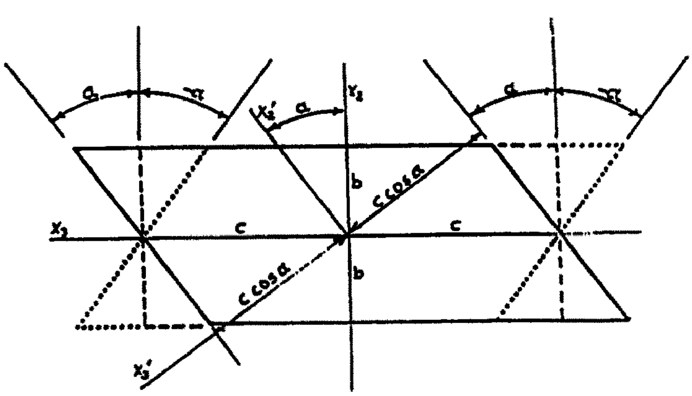

Fig. 3. Cross sections of strips.

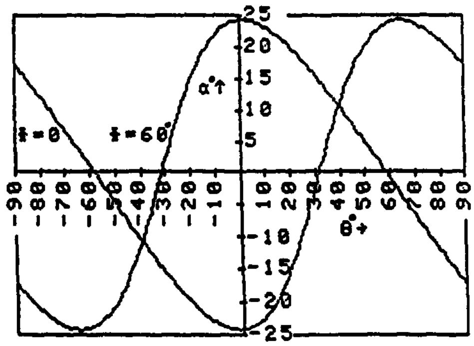

Fig. 4. Variation of dihedral angles, α, between face and edge planes of strip for $c_{pq}(0, \theta)$ and $c_{pq}(60°, -\theta)$ as functions of θ.

When $c_{pq}(0, \theta)$ changes to $c_{pq}(60°, -\theta)$, the frequencies do not change as c_{56} enters as c_{56}^2; but the mode-shape (52) changes and α, in (55), is reversed in sign so that the cross section changes, as illustrated in Fig. 3. The values of ±α for the full range of values of θ are illustrated in Fig. 4.

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